INDEFINITE INTEGRAL

15/06/2023

DATE
PAGE

BASIC FORMULAE

1. $\int x^n \, dx = \frac{x^{(n+1)}}{(n+1)} + C, \quad n \neq -1$

2. $\int \frac{1}{x} \, dx = \ell(|x|) + C$

3. $\int e^x \, dx = e^x + C$

4. $\int a^x \, dx = \frac{a^x}{\ell(a)} + C$

5. $\int cx \, dx = sx + C$

6. $\int sx \, dx = -cx + C$

7. $\int sec^2(x) \, dx = tx + C$

8. $\int cosec(x) \, cot(x) \, dx = -cosec(x) + C$

9. $\int sec(x) \, tx \, dx = sec(x) + C$

10. $\int cosec^2(x) \, dx = -cot(x) + C$

11. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = s^{-1}\left(\frac{x}{a}\right) + C$

12. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} t^{-1}\left(\frac{x}{a}\right) + C$

13. $\int \frac{1}{|x|\sqrt{x^2 - 1}} \, dx = sec^{-1}(x) + C$

(I) TRANSFORMATION / SIMPLIFICATION

Q. (i) $\int \frac{dx}{s_x^2 \; c_x^2}$

(ii) $\int \frac{s_x^6 + c_x^6}{s_x^2 c_x^2} \, dx$

(iii) $\int \frac{c_x - c_{2x}}{1 - c_x} \, dx$

(iv) $\int 5^{l(x)} \, dx$

A (i) $\int \frac{4 \, dx}{s_{2x}^2} = 4 \int \operatorname{cosec}^2(2x) \, dx = \underline{-\cot(2x)}$

(ii) $\int \frac{(s^2+c^2)(s^4 - s^2c^2 + s^4)}{s^2c^2} \, dx = \int \frac{1 - 3s^2c^2}{s^2c^2} \, dx$

$= \int 4 \operatorname{cosec}^2(2x) - 3 \; dx$

$= \underline{-\cot(2x) - 3x}$

(iii) $\int \frac{-2c^2 + c + 1}{1 - c} \, dx = \int 2c + 1 \; dx = \underline{2s_x + x}$

(iv) $\int x^{l(5)} \, dx = \underline{\frac{x^{l(5e)}}{l(5e)}}$

(II) SUBSTITUTION

Standard subs —

1. x^2+a^2 or $\sqrt{x^2+a^2}$ $\Rightarrow$ $x = a\,t\theta$ or $a.\cot(\theta)$

2. x^2-a^2 or $\sqrt{x^2-a^2}$ $\Rightarrow$ $x = a\sec(\theta)$ or $a\,\mathrm{cosec}(\theta)$

3. a^2-x^2 or $\sqrt{a^2-x^2}$ $\Rightarrow$ $x = a\sin(\theta)$ or $a\cos(\theta)$

Q (i) $\int \frac{s(lx)}{x}\,dx$ (ii) $\int \frac{3s+4c}{4s-3c}\,dx$

(iii) $\int \frac{8x+13}{\sqrt{4x+7}}\,dx$ (iv) $\int \frac{(x^2-1)}{(x^4+3x^2+1)\,t^{-1}\left(x+\frac{1}{x}\right)}\,dx$

A. (i) $u = lx$ $\Rightarrow$ $\int su\,du = -cu = \underline{-c_{l(x)}}$

$du = \frac{1}{x}\,dx$

(ii) $u = 4s-3c$ $\Rightarrow$ $\int \frac{du}{u} = l(u) = \underline{l(4s-3c)}$

$du = 4c+3s\;dx$

(iii) $u^2 = 4x+7$ $\Rightarrow$ $\int \frac{2u^2-1}{u}\left(\frac{u}{2}\,du\right) = \int u^2-\frac{1}{2}\,du$

$2u\,du = 4\,dx$

$= \frac{u^3}{3} - \frac{u}{2} = \underline{\frac{(4x+7)^{3/2}}{3} - \frac{(4x+7)^{1/2}}{2}}$

(iv) $\int \frac{\left(1-\frac{1}{x^2}\right)\,dx}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\,t^{-1}\left(\frac{x+1}{x}\right)} = \int \frac{du}{t^{-1}(u)\,(1+u^2)} = l(t^{-1}u)$

$= \underline{l\left(t^{-1}\left(x+\frac{1}{x}\right)\right)}$

$u = x+\frac{1}{x}$ $\Rightarrow$ $du = 1-\frac{1}{x^2}\,dx$

(III) INTEGRATION BY PARTS

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Choose u preferably using ILATE Rule

dv should be easily integrable.

Q (i) $\int (l(x))^2 \, dx$ (ii) $\int \frac{s^{-1}\sqrt{x} - c^{-1}\sqrt{x}}{s^{-1}\sqrt{x} + c^{-1}\sqrt{x}} \, dx$

(iii) $\int e^{ax} \, S\, bx \, dx$ (iv) $\int e^{ax} \, C\, bx \, dx$

A (i) $dv = dx \Rightarrow v = x$

$u = (l(x))^2 \Rightarrow du = \frac{2 l(x)}{x} \, dx$

$$= x\, l^2(x) - \int x \left(\frac{2 l(x)}{x}\right) dx = x\, l^2(x) - 2 \int l(x) \, dx$$

$$= x l^2(x) - 2\left[x l(x) - \int dx \right]$$

$dv = dx \Rightarrow v = x$

$u = l(x) \Rightarrow du = \frac{dx}{x}$

$$= x^2 l(x) - 2x\, l(x) + 2x$$

(ii) $u^2 = x \Rightarrow 2u \, du = dx \Rightarrow \int \frac{s^{-1}_u - c^{-1}_u}{s^{-1}_u + c^{-1}_u} (2u \, du)$

$$\Rightarrow \frac{2}{\pi} \int (s^{-1}_u - c^{-1}_u)(2u \, du)$$

$dv = 2u \, du \Rightarrow v = u^2$

$z = s^{-1}_u - c^{-1}_u \Rightarrow dz = \frac{2 \, du}{\sqrt{1-u^2}}$

$$= \frac{2}{\pi}\left(u^2 (s^{-1}_u - c^{-1}_u) - \int \frac{2u^2 \, du}{\sqrt{1-u^2}} \right.$$

DATE ______
PAGE ______

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$\int \frac{2u^2}{1-u^2} du = \int \frac{2\sin^2\theta}{\cos\theta} \cos\theta \, d\theta = \int 1 - \cos 2\theta \, d\theta$$

$$= \theta - \sin\theta = \underline{\sin^{-1}u - u}$$

$$= \frac{2}{\pi}\left(u^2(\sin^{-1}u - \cos^{-1}u) - \sin^{-1}u + u \right)$$

$$= \frac{2}{\pi}\left(x(\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}) - \sin^{-1}\sqrt{x} + \sqrt{x} \right)$$

__Remember!__

(iii) $dv = e^{ax} dx \Rightarrow v = \frac{e^{ax}}{a} \Rightarrow \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx \, dx$

$u = \sin bx \Rightarrow du = b\cos bx \, dx \Rightarrow \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left(\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx \, dx \right)$

$dv = e^{ax} dx \Rightarrow v = \frac{e^{ax}}{a} \Rightarrow I = \frac{e^{ax}\sin bx}{a} - \frac{b e^{ax}\cos bx}{a^2} - \frac{b^2 I}{a^2}$

$u = \cos bx \Rightarrow du = -b\sin bx \, dx \Rightarrow I = \frac{e^{ax}}{(a^2+b^2)}\left[a\sin bx - b\cos bx \right]$

__Remember!__

(iv) $$I = \frac{e^{ax}}{(a^2+b^2)}\left[a\cos bx + b\sin bx \right]$$

DATE
PAGE

NOTE: $$\boxed{\int e^x (f(x) + f'(x))\, dx = e^x f(x) + C}$$

Q. (i) $\int e^x \left(\frac{1+\sin x \cos x}{\cos^2 x}\right) dx$

(ii) $\int e^{2x} \left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

(iii) $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

(iv) $\int e^x \left(\frac{x^2+3x+2}{(x+2)^3}\right) dx$

A (i) $\int e^x \cdot (\tan x + \sec^2 x)\, dx = \underline{e^x \tan x}$

(ii) $u = 2x$ $\Rightarrow \frac{1}{2}\int e^u \left(\frac{1+\sin u}{1+\cos u}\right) du$
$du = 2dx$

$\Rightarrow \frac{1}{2}\int e^u \left[\tan\left(\frac{u}{2}\right) + \frac{1}{2}\sec^2\left(\frac{u}{2}\right)\right] du$

$= \frac{1}{2} e^u \tan\left(\frac{u}{2}\right) = \underline{\frac{e^{2x}\tan x}{2}}$

(iii) $\int e^x \left(\frac{1+x^2-2x}{(1+x^2)^2}\right) dx = \int e^x \left(\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right) dx$

$= \left(\frac{e^x}{1+x^2}\right)$

DATE
PAGE

(iv) $\int e^x \frac{x+1}{(x+2)^2} dx = \int e^x \left[\frac{1}{(x+2)} - \frac{1}{(x+2)^2}\right] dx$

$= \left(\frac{e^x}{x+2}\right)$

• Indirect Sub^n –

Q. $\int \frac{s+c}{9+16s2x} dx = \int \frac{(s+c)}{25-16(s-c)^2} dx = \int \frac{-du}{25-16u^2}$

$= \frac{1}{10} \int \frac{1}{5+4u} + \frac{1}{5-4u} du$

$u = s-c$

$du = (c+s)dx$

$= \frac{1}{40} l\left(\frac{5+4u}{5-4u}\right)$

$= \frac{1}{40} l\left(\frac{5+4s-4c}{5-4s+4c}\right)$

DATE________
PAGE________

STANDARD FORMULAE

1. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$$

2. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

3. $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

4. $$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$$

5. $$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

6. $$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} s^{-1}\left(\frac{x}{a}\right) + C$$

Q. (i) $\int \frac{1}{\sqrt{x^2-2x+3}}\, dx$ (ii) $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\, dx$

☆ (iii) $\int \sqrt{\frac{x}{a^3-x^3}}\, dx$ ☆ (iv) $\int \sqrt{\frac{s(x-\alpha)}{s(x+\alpha)}}\, dx$

DATE
PAGE

A (i) $$\int \frac{1}{\sqrt{(x-1)^2+2}}\,dx = \ln\left|(x-1)+\sqrt{(x-1)^2+2}\right|$$

(ii) $$\int \frac{e^x\,dx}{\sqrt{9-(e^x+2)^2}} = \frac{1}{3}\sin^{-1}\left(\frac{e^x+2}{3}\right)$$

★ (iii) $u^2 = x^3$

$2u\,du = 3x^2\,dx$

$\Rightarrow dx = 2u^{-1/3}\,du$

$$\Rightarrow \int \frac{\sqrt{u^{2/3}}}{\sqrt{a^3-u^2}}\,2u^{-1/3}\,du$$

$$= 2\int \frac{1}{\sqrt{a^3-u^2}}\,du$$

$$= \frac{2}{a^{3/2}}\sin^{-1}\left(\frac{u}{a^{3/2}}\right) = \frac{2}{a^{3/2}}\,s^{-1}\left(\left(\frac{x}{a}\right)^{3/2}\right)$$

★ (iv) $$\int \frac{s(x-\alpha)}{\sqrt{s(x+\alpha)\,s(x-\alpha)}}\,dx = \int \frac{s_x c_\alpha - c_x s_\alpha}{\sqrt{s_x^2-s_\alpha^2}}\,dx$$

$$= \int \frac{s_x c_\alpha}{\sqrt{s_x^2-s_\alpha^2}} - \frac{c_x s_\alpha}{\sqrt{s_x^2-s_\alpha^2}}\,dx$$

$$= -\int \frac{(-s_x)c_\alpha\,dx}{\sqrt{c_\alpha^2-c_x^2}} - \int \frac{c_x s_\alpha\,dx}{\sqrt{s_x^2-s_\alpha^2}}$$

$$= -c_\alpha\,s^{-1}\left(\frac{c_x}{c_\alpha}\right) - s_\alpha\,l\left|s_x+\sqrt{s_x^2-s_\alpha^2}\right|$$

21/06/2023

DATE

PAGE

STANDARD FORMS

→ Type I

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad \int \sqrt{ax^2+bx+c}\, dx$$

Complete the sq. & use standard formulae.

→ Type II

$$\int \frac{px+q}{(ax^2+bx+c)}\, dx, \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx, \quad \int (px+q)\sqrt{ax^2+bx+c}\, dx$$

Rewrite $px+q = \lambda \underbrace{(2ax+b)}_{\frac{d}{dx}(ax^2+bx+c)} + \mu$

& substitute.

Q. $\displaystyle\int \frac{2s2x - cx}{6 - c^2x - 4sx}\, dx$

A. $\displaystyle\int \frac{4sxcx - cx}{s^2x - 4sx + 5}\, dx = \int \left(\frac{4s-1}{s^2-4s+5}\right) c\, dx$

$\displaystyle = 4\int \left(\frac{s-2}{(s-2)^2+1}\right) c\, dx + \int \frac{7c}{(s-2)^2+1}\, dx$

$= 2\, l|s^2-4s+5| + 7\, t^{-1}(s-2)$

DATE ______
PAGE ______

→ Type III

$$\int \frac{ax^2+bx+c}{px^2+qx+r}\,dx, \quad \int \frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}\,dx, \quad \int (ax^2+bx+c)\sqrt{px^2+qx+r}\,dx$$

Rewrite $ax^2+bx+c = \lambda(px^2+qx+r) + \mu\underbrace{(2px+q)}_{\frac{d}{dx}(px^2+qx+r)} + r$

Q. $$\int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$$

A. $2x^2+5x+4 = 2(x^2+x+1) + \frac{3}{2}(2x+1) + \frac{1}{2}$

$$\Rightarrow \int 2\sqrt{x^2+x+1} + \frac{3}{2}\frac{(2x+1)}{\sqrt{x^2+x+1}} + \frac{1}{2}\frac{1}{\sqrt{x^2+x+1}}\,dx$$

→ Type IV

$$\int \frac{1}{ac^2_x+bs^2_x}\,dx, \quad \int \frac{1}{a+bs^2_x}\,dx, \quad \int \frac{1}{a+bc^2_x}\,dx,$$

$$\int \frac{1}{(as_x+bc_x)^2}\,dx, \quad \int \frac{1}{a+bs^2_x+cc^2_x}\,dx$$

Divide num. & den. by $\cos^2(x)$ &
substitute $u = t_x$

Q. (i) $\int \frac{1}{4s_x^2 + 9c_x^2} dx$ (ii) $\int \frac{s_x}{s_{3x}} dx$

A. (i) $\int \frac{\sec^2(x)\, dx}{4\left(t_x^2 + \frac{9}{4}\right)} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) t^{-1}\left(\frac{2t_x}{3}\right)$

$= \frac{1}{6} t^{-1}\left(\frac{2t_x}{3}\right)$

(ii) $\int \frac{1}{3 - 4s_x^2} dx = \int \frac{\sec^2(x)\, dx}{3\sec^2 - 4t^2}$

$= \int \frac{\sec^2(x)\, dx}{3 - t^2} = -\int \frac{\sec^2(x)\, dx}{t^2 - 3}$

$= -\frac{1}{2\sqrt{3}}\, l\left|\frac{t - \sqrt{3}}{t + \sqrt{3}}\right|$

→ Type V

$\int \frac{1}{a s_x + b c_x} dx$, $\int \frac{1}{a + b s_x} dx$, $\int \frac{dx}{a + b c_x}$,

$\int \frac{dx}{a s_x + b c_x + c}$

Convert s_x & c_x into $t_{x/2}$ & substitute $u = t_{x/2}$

DATE
PAGE

Q. $\int \frac{dx}{s_x + c_x + 2} = \int \frac{dx}{\frac{2t + 1-t^2}{1+t^2} + 2}$

$= 2\int \frac{1/2 \sec^2(x/2)\ dx}{2t+1-t^2+2+2t^2}$

$= 2\int \frac{\frac{1}{2}\sec^2(x/2)\ dx}{(t+1)^2+2} = \sqrt{2}\ t^{-1}\left(\frac{t_{x/2}+1}{\sqrt{2}}\right)$

→ Type $\underline{VI}$

$\int \frac{pc_x + qs_x + r}{ac_x + bs_x + c}\ dx$, $\int \frac{pc_x + qs_x}{ac_x + bs_x}\ dx$

Rewrite $\underbrace{N}_{\text{(numerator)}} = \lambda D + \mu D' + \gamma$
└ (denominator)

Q. $\int \frac{2+3c_x}{s_x+2c_x+3}\ dx$

A. $2+3c_x = \lambda(s_x+2c_x+3) + \mu(c_x - 2s_x) + \gamma$

$= (\lambda - 2\mu)s_x + (2\lambda+\mu)c_x + (3\lambda+\gamma)$

$\lambda = 2\mu$, $2\lambda+\mu = 3$, $3\lambda+\gamma = 2$

$\Rightarrow$ $\underline{\mu = 3/5}$, $\underline{\lambda = 6/5}$, $\underline{\gamma = -8/5}$

DATE
PAGE

→ Type VII

$$\int \frac{N(x)}{D(x)}\,dx \quad , \quad deg(N(x)) \geq D(x)$$

Rewrite $N(x) = Q(x)\,D(x) + R(x)$; $deg(R(x)) < N(x)$

$$\Rightarrow \int Q(x) + \frac{R(x)}{D(x)}\,dx$$

→ Type VIII (Partial Fraction Decomposition)

$$\int \frac{N(x)}{D(x)}\,dx \quad , \quad deg(D(x)) > N(x)$$

eg- (i) $\dfrac{1}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-3)}$

(ii) $\dfrac{x+5}{(x-1)(x+2)^2} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x+2)^2}$

(iii) $\dfrac{2x+1}{(3x+2)(4x^2+5x+6)} = \dfrac{A}{(3x+2)} + \dfrac{Bx+C}{\underbrace{(4x^2+5x+6)}_{D}}$ ← {General polynomial of $deg(D)-1$}

(iv) $\dfrac{2x^4+2x^2+x+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

DATE________
PAGE________

Q. $\displaystyle\int \frac{1-x\ln x}{x\left(1-(xe^{\ln x})^3\right)}\,dx$

A. $u = xe^{\ln x} \Rightarrow du = e^{\ln x}(1-x\ln x)\,dx$

$$\Rightarrow \int \frac{du}{u(1-u^3)} = -\int \frac{du}{u(u-1)(u^2+u+1)}$$

$$\frac{1}{u(u-1)(u^2+u+1)} = \frac{A}{u} + \frac{B}{(u-1)} + \frac{Cu+D}{(u^2+u+1)}$$

$$\Rightarrow A(u-1)(u^2+u+1) + Bu(u^2+u+1) + (Cu+D)(u-1)(u) = 1$$

$u=0$ $-A = 1 \Rightarrow A = -1$

$u=1$ $3B = 1 \Rightarrow B = 1/3$

$u=-1$ $-2A - B + 2D - 2C = 1 \Rightarrow C - D = 1/3$

$u=2$ $7A + 14B + 4C + 2D = 1$

$\Rightarrow -7 + \frac{14}{3} + 4C + 2D = 1 \Rightarrow 2C + D = 5/3$

$\Rightarrow C = 2/3$, $D = 1/3$

$$\Rightarrow -\int \frac{-1}{u} + \frac{1}{3(u-1)} + \frac{1}{3}\frac{(2u+1)}{(u^2+u+1)}\,du$$

$$= -\left(-\ell|u| + \frac{1}{3}\ell|u-1| + \frac{1}{3}\ell|u^2+u+1|\right)$$

$$= \ell|xe^{\ln x}| - \frac{1}{3}\ell|xe^{\ln x}-1| - \frac{1}{3}\ell|x^2e^{2\ln x} + xe^{\ln x} + 1|$$

→ Derived Sub^n (Twin Problems)

(A) Algebraic twins :

$$\int \frac{2x^2}{x^4+1}\,dx \quad , \quad \int \frac{2}{x^4+1}\,dx \quad , \quad \int \frac{2x^2}{x^4+kx^2+1}\,dx$$

(B) Trigonometric twins :

$$\int \sqrt{\tan x}\,dx \quad , \quad \int \sqrt{\cot x}\,dx \quad , \quad \int \frac{1}{\sin^4 x + \cos^4 x}\,dx \quad ,$$

$$\int \frac{1}{\sin^6 x + \cos^6 x}\,dx \quad , \quad \int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos x}\,dx$$

Q. $$\int \frac{2x^2}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}\,dx + \int \frac{x^2-1}{x^4+1}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx + \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1}{\left(x-\frac{1}{x}\right)^2+2}\,d\left(x-\frac{1}{x}\right) + \int \frac{1}{\left(x+\frac{1}{x}\right)^2-2}\,d\left(x-\frac{1}{x}\right)$$

Q. $$\int \sqrt{\tan x}\,dx$$

A. $u^2 = \tan x$

$\Rightarrow 2u\,du = \sec^2(x)\,dx$

$= 1+u^4\,dx$

$$\Rightarrow \int \frac{2u^2}{1+u^2}\,du$$

DATE
PAGE

Q. $\int \frac{1}{s^6 x + c^6 x} dx$

A. $\int \frac{1}{s^4 - s^2c^2 + c^4} dx = \int \frac{\sec^4 dx}{t^4 - t^2 + 1} = \int \frac{1+t^2}{t^4-t^2+1} \sec^2(x) dx$

$= \frac{1}{2} \int \frac{1}{t^2+t+1} + \frac{1}{t^2-t+1} d(tx)$

$= \frac{1}{\sqrt{3}} t^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} t^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)$

Twin method : $\int \frac{1+t^2}{t^4-t^2+1} d(tx) = \int \frac{\left(1+\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2} - 1} d(tx)$

$= \int \frac{1}{\left(t-\frac{1}{t}\right)^2 + 1} d\left(t - \frac{1}{t}\right)$

$= t^{-1}\left(t - \frac{1}{t}\right)$

22/06/2023

→ $\int s^m_x \, c^n_x \, dx$

① $m, n \in \mathbb{N}$

1.1 One of them is odd,

⇒ Substitute for even power

1.2. Both odd,

⇒ Substitute either of them.

1.3 Both even,

⇒ use Trigonometric Identities

DATE

PAGE

(2) $m, n \in \mathbb{Q}$ & $\left(\frac{m+n-2}{2}\right)$ is a (-ve) integer

$\Rightarrow$ Substitute $u = \cot(x)$ or tx as per suitability.

Q. (i) $\int s^3 c^5 \, dx$ (ii) $\int s^{-11/3} c^{-1/3} \, dx$

A. (i) $u = sx$

$du = cx \, dx$

$\Rightarrow \int u^3 (1-u^2)^2 \, du$

$= \int u^3 - 2u^5 + u^7 \, du$

$= \frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{5} = \frac{s_x^4}{4} - \frac{s_x^6}{3} + \frac{s_x^8}{8}$

(ii) $\int t^{-11/3} \overset{(sec^2 x)^{-5/3}}{c^{10/3}} \, dx$

$u = tx$

$du = sec^2(x) \, dx$

$= \int u^{-11/3} (1+u^2)^{-5/3} \frac{du}{(1+u^2)}$

$= \int u^{-11/3} (1+u^2)^{-8/3} \, du = \int \frac{1}{u(u+u^3)^{8/3}} \, du$

$\int \frac{t^{1/3}}{s^4} \, dx = \int \frac{-(1+\cot^2) \quad -cosec^2(x) \, dx}{(\cot(x))^{1/3}}$

$u = \cot(x)$

$du = -cosec^2(x) \, dx$

$= \int -\left(\frac{1+u^2}{u^{1/3}}\right) du$

$= \int -u^{-1/3} - u^{5/3} \, du = -\frac{3u^{2/3}}{2} - \frac{3}{8} u^{8/3}$

$= \frac{-3}{2} \cot^{2/3}(x) - \frac{3}{8} \cot^{8/3}(x)$

DATE
PAGE

→ Irrational Algebraic $f x^n s$

1. $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ $\Rightarrow$ put $u = cx+d$

2. $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ $\Rightarrow$ put $u = px+q$

3. $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ $\Rightarrow$ put $\frac{1}{u} = ax+b$

4. $\int \frac{dx}{(ax^2+b)(cx^2+d)}$ $\Rightarrow$ put $u = \frac{1}{x}$

5. $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$ $\Rightarrow$ put $\frac{1}{u} = (x-k)$

$r \geq 2, \quad r \in \mathbb{Z}$

6. $\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}}\,dx$ $\Rightarrow$ rewrite ax^2+bx+c

$= A(dx+e)(2fx+g) + B(dx+e) + C$

→ Red^n formulae

1. If $I_n = \int (l(x))^n \, dx$, then $\boxed{I_n + n I_{(n-1)} = x (l(x))^n}$

Proof :- (IBP) $\int l^n \cdot 1 \, dx = x l^n - n \int l^{(n-1)} \left(\frac{1}{x}\right)(x) \, dx$

$\Rightarrow I_n + n I_{n-1} = x l^n$

2. If $I_n = \int s^n \, dx$, then $\boxed{I_n - \left(\frac{n-1}{n}\right) I_{(n-2)} = \frac{-s^{(n-1)} c}{n}}$

3. If $I_n = \int c^n \, dx$, then $\boxed{I_n - \left(\frac{n-1}{n}\right) I_{(n-2)} = \frac{c^{(n-1)} s}{n}}$

4. If $I_n = \int t^n \, dx$, then $\boxed{I_n + I_{(n-2)} = \frac{t^{(n-1)}}{(n-1)}}$

5. If $I_n = \int \text{cosec}^n(x) \, dx$, then $\boxed{I_n - \left(\frac{n-2}{n-1}\right) I_{(n-2)} = - \frac{\text{cosec}^{(n-2)} \cot}{(n-1)}}$

6. If $I_n = \int \sec^n(x) \, dx$, then $\boxed{I_n - \left(\frac{n-2}{n-1}\right) I_{(n-2)} = \frac{\sec^{(n-2)} t}{(n-1)}}$

7. If $I_n = \int \cot^n(x) \, dx$, then $\boxed{I_n + I_{(n-2)} = \frac{-\cot^{(n-1)}}{(n-1)}}$

8. If $I_{(m,n)} = \int c_x^m \, s_{nx} \, dx$,

then $\boxed{I_{(m,n)} - \left(\frac{m}{m+n}\right) I_{(m-1,\, n-1)} = \frac{-c_x^m \, c_{nx}}{m+n}}$

DATE
PAGE

Q1. If $I_{(n,m)} = \int \frac{s^n}{c^m}\,dx$, P.T

$$I_{(n,m)} + \left(\frac{n-1}{m-1}\right) I_{(n-2,m-2)} = \left(\frac{1}{m-1}\right)\frac{s^{(n-1)}}{c^{(m-1)}}$$

Q2. For $m, n \in \mathbb{N}$, $x > 0$, evaluate $\int (x^{3m} + x^{2n} + x^{m})(2x^{2m} + 3x^{m} + 6)^{1/m}\,dx$

Q3. Evaluate $\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{l(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\,dx$

A 1. $\int \underbrace{s^{(n-1)}}_{u} \underbrace{\frac{s}{c^m}\,dx}_{dv} = \frac{1}{(m-1)}\frac{s^{(n-1)}}{c^{(m-1)}} - \frac{(n-1)}{(m-1)} \int \frac{s^{(n-2)}}{c^{(m-2)}}\,dx$

$$\Rightarrow I_{(n,m)} + \left(\frac{n-1}{m-1}\right) I_{(n-2,m-2)} = \frac{1}{(m-1)}\frac{s^{(n-1)}}{c^{(m-1)}}$$

2. $\int \left(x^{(3m-1)} + x^{(2m-1)} + x^{(m-1)}\right)\left(2x^{3m} + 3x^{2m} + 6x^{m}\right)^{1/m} dx$

$u = 2x^{3m} + 3x^{2m} + 6x^{m}$ $\Rightarrow \int \frac{1}{6} u^{1/m}\,du$

$du = 6\left(x^{(3m-1)} + x^{(2m-1)} + x^{(m-1)}\right) dx$

$\Rightarrow \frac{m}{6(m+1)}\left(2x^{3m} + 3x^{2m} + 6x^{m}\right)^{\left(\frac{m+1}{m}\right)}$

3 $u = x^{1/2}$ $\Rightarrow \int\left(\frac{1}{u^3+u^4} + \frac{l(1+u^2)}{u^4+u^6}\right) 12u^{11}\,du$

$\Rightarrow 12\int \underbrace{\frac{u^8}{u+1}}_{\text{Partial Fraction}}\,du + 12\int \frac{u^7\, l(1+u^2)}{(1+u^2)}\,du$

$= 12\int \underbrace{(u^5 - 3u^3 + u)\, l(1+u^2)\,du}_{v = 1+u^2 \Rightarrow \text{By Parts}} - 12\int \underbrace{\frac{u\, l(1+u^2)\,du}{(1+u^2)}}_{v = l(1+u^2)}$